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Fractional Nambu Mechanics

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Abstract The fractional generalization of Nambu mechanics is constructed by using the differential forms and exterior derivatives of fractional orders. The generalized Pfaffian equations are obtained and one example is investigated in details.

Keywords Fractional derivative · Hamiltonian mechanics · Nambu mechanics · Fractional differential forms

1 Introduction

Fractional calculus deals is as old as the classical calculus and it deals with derivative and integrals of any order [8–11, 13, 14, 21, 23, 27]. Derivatives and integrals of fractional order have found many applications in the recent studies in science and engineering [1, 2, 4, 16, 24–26, 30, 31]. For example, it includes chaotic dynamics [32, 33] mechanics of fractal media [5, 19, 29], quantum mechanics [12, 17] and etc.

This formalism is shown to provide a suitable framework for the description of non-integrable systems [22]. Several authors have fractionized the Hamiltonian systems and applied it to several systems [25, 26, 31]. For example, Riewe has shown that Lagrangian with

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fractional derivative lead directly to equations of motion with non conservative classical forces such as friction.

In 1973 Nambu proposed a generalization of ordinary Hamiltonian mechanics which is now called Nambu mechanics. Nambu systems were investigated by several author [6, 7, 15, 18, 20, 22, 28]. The connexion between Nambu and Killing-Yano tensors was given in [3].

In this paper we have shown that fractional Nambu systems can be proposed as generalization of the fractional Hamiltonian systems.

Section 2 contains a brief summery of fractional derivative, especially Caputo derivative.

In Sect. 3 we explain briefly the classical Nambu mechanics.

In Sect. 4 we obtain the fractional Hamiltonian equations from the fractional form by use of generalized Pfaffian equations.

In Sect. 5 we obtain fractional Nambu mechanic from the fractional generalization of the above mentioned 2-form and we show that fractional Nambu mechanic can be proposed as generalization of the usual fractional Hamiltonian systems. All of these equations were found consistent with Liouville theorem [30]. One example in investigated in details.

Section 7 contains the conclusions.

2 Basic Tools

The fractional derivative has different definition [23]. The classical definition is called Riemann Liouville derivative. Due to reasons, concerning the initial and boundary conditions, it is more convenient to use the Caputo fractional derivative. Its main advantage is that the initial condition take the same form as for integer-order differential equations. The Caputo fractional derivative is defined by

$$D_x^\alpha f(x) = {}_0^C D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{1}{(x-\tau)^{-n+\alpha+1}} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{1}$$

where α is order of derivative such that $n-1 \le \alpha < n$.

Using $D_{x_k}^\alpha x_k = \Gamma(2-\alpha) {x_k}^{\alpha-1}$ and the definition

$$\mathbf{D}_{x_k}^\alpha = \{D_{x_k}^\alpha x_k\}^{-1} D_{x_k}^\alpha = \Gamma(2-\alpha) x_k^{\alpha-1} D_{x_k}^\alpha,$$

we get $\mathbf{D}_{x_k}^\alpha x_k = 1$ [30]. Using the rule

$$\mathbf{D}_{x_k}^\alpha (fg) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (\mathbf{D}_{x_k}^{\alpha-k} f) \frac{\partial^k g}{\partial x^k}$$

and the relation

$$\frac{\partial^k}{\partial x^k}\{(dx)^\alpha\} = 0,$$

for example, we have

$$d^\alpha\{A^i (dx_i)^\alpha\} = \{\mathbf{D}_{x_j}^\alpha A^i\}(dx_j)^\alpha \wedge (dx_i)^\alpha$$

x_j is arbitrary variables.

3 Classical Nambu Mechanics

In 1973 Nambu proposed a generalization of ordinary Hamiltonian mechanics [18] which is called Nambu mechanics. In this section we study Nambu mechanics in terms of exterior differential form [20].

Here a Nambu bracket, which is a generalization of the usual Poisson bracket is explained and we outline the canonical formulation of mechanics by use of the Poincare-Cartan 1-form $\Omega^{(1)}$,

$$\Omega^{(1)} = pdq - H(p,q)dt, \tag{2}$$

and we introduce 2-form for Nambu mechanics: for three-dimensional case it is, $\Omega^{(2)}$

$$\Omega^2 = qdp \wedge dr - H_1 dH_2 \wedge dt.$$

In standard analytical, the equations of motion are

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q}. \end{aligned} \tag{3}$$

The temporal development of any function $f(p,q)$ can be written in terms of the Poisson bracket as

$$\frac{df}{dt} = \{f, H\}. \tag{4}$$

The Poisson bracket $\{.,.\}$ can also be written in terms of the Jacobian [6]

$$\{f, g\} = \frac{\partial(f, H)}{\partial(p,q)}. \tag{5}$$

Noticing this form, Nambu generalized formally the equations of motion (3) to

$$\frac{df}{dt} = \{f, H_1, H_2\} = \frac{\partial(f, H_1, H_2)}{\partial(q,p,r)}, \tag{6}$$

where p, q and r denote a triplet of dynamical variables and H_1, H_2 are two Hamiltonian with argument q, p and r. Equation (6) is a natural generalization of (4) and extended Poisson bracket $\{.,.,.\}$ is called the Nambu bracket.

Substituting the triplet (p,q,r) for f respectively in (6) we have

$$\begin{aligned} \dot{q} &= \frac{\partial(q, H_1, H_2)}{\partial(q,p,r)} = \frac{\partial(H_1, H_2)}{\partial(p,r)}, \\ \dot{p} &= \frac{\partial(p, H_1, H_2)}{\partial(q,p,r)} = \frac{\partial(H_1, H_2)}{\partial(r,q)}, \\ \dot{r} &= \frac{\partial(r, H_1, H_2)}{\partial(q,p,r)} = \frac{\partial(H_1, H_2)}{\partial(q,p)}. \end{aligned} \tag{7}$$

Now we outline Hamiltonian mechanics in terms of exterior differential form as preparation for treating Nambu mechanics. Let us begin by considering the following 1-form $\Omega^{(1)}$on

R^{n+1} with coordinate q, p and time

$$\Omega^{(1)} = pdq - H(p,q)dt,$$

the exterior differential of $\Omega^{(1)}$ is [20]

$$d\Omega^{(1)} = \theta \wedge \rho,$$

where

$$\begin{aligned} \theta &= dp + \frac{\partial H}{\partial q} dt, \\ \rho &= dq - \frac{\partial H}{\partial q} dt. \end{aligned} \tag{8}$$

This indicate the close relation between the fundamental 1-form and the Hamiltonian equations of motion because the Pfaffian equations $\theta = 0$ and $\rho = 0$ are exactly the Hamiltonian equations (3). Now we will restrict our selves to three dimensional case. The fundamental 1-form in Hamiltonian mechanics can be generalized to the following 2-form $\Omega^{(2)}$ on R^4 of dynamical variables q, p, r and a time t,

$$\Omega^2 = qdp \wedge dr - H_1 dH_2 \wedge dt. \tag{9}$$

The reason why we choose the above 2-form in Nambu mechanics is shown as follows. The differential of $\Omega^{(2)}$ is written as

$$d\Omega^{(2)} = \theta \wedge \rho \wedge \sigma,$$

where

$$\begin{aligned} \theta &= dq - \frac{\partial(H_1, H_2)}{\partial(p, r)} dt, \\ \rho &= dr - \frac{\partial(H_1, H_2)}{\partial(r, q)} dt, \\ \sigma &= dr - \frac{\partial(H_1, H_2)}{\partial(q, p)} dt. \end{aligned}$$

Now the Pfaffian equations $\theta = 0, \rho = 0, \sigma = 0$ are equivalent to Nambu equations [20].

4 Fractional Hamiltonian Mechanics

The form

$$\Omega^{(1)} = pdq - H(p,q)dt, \tag{10}$$

is called the Poincare-Cartan 1-form. The fractional generalization of this form can be defined by

$$\Omega^{(1)}_\alpha = p(dq)^\alpha - H(p,q)(dt)^\alpha. \tag{11}$$

Note that $\Omega_\alpha^{(1)}$ is a fractional 1-form that can be called a fractional Poincare-Cartan 1-form. Using

$$d^\alpha p = \mathbf{D}_t^\alpha p(dt)^\alpha + \mathbf{D}_p^\alpha p(dp)^\alpha + \mathbf{D}_q^\alpha p(dq)^\alpha$$

and identities

$$\begin{aligned}
&d^\alpha H \wedge (dt)^\alpha = \mathbf{D}_t^\alpha H(dt)^\alpha \wedge (dt)^\alpha + \mathbf{D}_p^\alpha H(dp)^\alpha \wedge (dt)^\alpha + \mathbf{D}_q^\alpha H(dq)^\alpha \wedge (dt)^\alpha,\\
&(dt)^\alpha \wedge (dt)^\alpha = 0, \qquad (dt)^\alpha \wedge (dp)^\alpha = -(dp)^\alpha \wedge (dt)^\alpha,\\
&\mathbf{D}_p^\alpha p = 1, \qquad \mathbf{D}_q^\alpha p = 0 \quad \text{and} \quad \mathbf{D}_t^\alpha p = 0,
\end{aligned}$$

we have

$$d\Omega_\alpha^{(1)} = \{(dp)^\alpha + \mathbf{D}_q^\alpha H(dt)^\alpha\} \wedge \{(dq)^\alpha - \mathbf{D}_p^\alpha H(dt)^\alpha\}.$$

Then by generalized Pfaffian equations, we get

$$\begin{aligned}
\frac{(dq)^\alpha}{(dt)^\alpha} &= \mathbf{D}_p^\alpha H,\\
\frac{(dp)^\alpha}{(dt)^\alpha} &= -\mathbf{D}_q^\alpha H,
\end{aligned} \tag{12}$$

which is called fractional Hamiltonian equations.

5 Fractional Nambu Mechanics

The fractional generalization of the 2-form (9) can be defined by

$$\Omega_\alpha^{(2)} = q(dp)^\alpha \wedge (dr)^\alpha - H_1 d^\alpha H_2 \wedge (dt)^\alpha. \tag{13}$$

The fractional exterior derivative of the fractional 2-form (13) is

$$d^\alpha \Omega_\alpha^{(2)} = d^\alpha(q(dp)^\alpha \wedge (dr)^\alpha) - d^\alpha(H_1 d^\alpha H_2 \wedge (dt)^\alpha). \tag{14}$$

Using identities

$$\begin{aligned}
&d^\alpha\{q(dp)^\alpha \wedge (dr)^\alpha\}\\
&\quad = \mathbf{D}_t^\alpha q(dt)^\alpha \wedge (dp)^\alpha \wedge (dr)^\alpha + \mathbf{D}_q^\alpha q(dq)^\alpha \wedge (dp)^\alpha \wedge (dr)^\alpha\\
&\qquad + \mathbf{D}_p^\alpha q(dp)^\alpha \wedge (dp)^\alpha \wedge (dr)^\alpha + \mathbf{D}_r^\alpha q(dr)^\alpha \wedge (dp)^\alpha \wedge (dr)^\alpha,
\end{aligned}$$

and

$$\begin{aligned}
d^\alpha H_2 &= \mathbf{D}_t^\alpha H_2(dt)^\alpha + \mathbf{D}_q^\alpha H_2(dq)^\alpha + \mathbf{D}_p^\alpha H_2(dp)^\alpha + \mathbf{D}_r^\alpha H_2(dr)^\alpha,\\
d^\alpha H_2 \wedge (dt)^\alpha &= \mathbf{D}_t^\alpha H_2(dt)^\alpha \wedge (dt)^\alpha + \mathbf{D}_q^\alpha H_2(dq)^\alpha \wedge (dt)^\alpha\\
&\quad + \mathbf{D}_p^\alpha H_2(dp)^\alpha \wedge (dt)^\alpha + \mathbf{D}_r^\alpha H_2(dr)^\alpha \wedge (dt)^\alpha,\\
d^\alpha H_1 &= \mathbf{D}_t^\alpha H_1(dt)^\alpha + \mathbf{D}_q^\alpha H_1(dq)^\alpha + \mathbf{D}_p^\alpha H_1(dp)^\alpha + \mathbf{D}_r^\alpha H_1(dr)^\alpha,
\end{aligned}$$

we get

$$d^\alpha \Omega_\alpha^{(2)} = (D_q^\alpha H_1 D_p^\alpha H_2 - D_p^\alpha H_1 D_q^\alpha H_2)(dq)^\alpha \wedge (dp)^\alpha \wedge (dt)^\alpha + (D_q^\alpha H_1 D_r^\alpha H_2 - D_r^\alpha H_1 D_q^\alpha H_2)(dq)^\alpha \wedge (dr)^\alpha \wedge (dt)^\alpha + (D_p^\alpha H_1 D_r^\alpha H_2 - D_r^\alpha H_1 D_p^\alpha H_2)(dp)^\alpha \wedge (dr)^\alpha \wedge (dt)^\alpha.$$

By the definition of

$$\frac{\partial^\alpha (H_1, H_2)}{\partial^\alpha (p, r)} = (\mathbf{D}_p^\alpha H_1 \mathbf{D}_r^\alpha H_2 - \mathbf{D}_r^\alpha H_1 D_p^\alpha H_2),$$
$$\frac{\partial^\alpha H_1, H_2)}{\partial^\alpha (q, r)} = (\mathbf{D}_q^\alpha H_1 D_r^\alpha H_2 - \mathbf{D}_r^\alpha H_1 D_q^\alpha H_2), \tag{15}$$
$$\frac{\partial^\alpha (H_1, H_2)}{\partial^\alpha (q, p)} = (\mathbf{D}_q^\alpha H_1 \mathbf{D}_p^\alpha H_2 - \mathbf{D}_p^\alpha H_1 \mathbf{D}_q^\alpha H_2),$$

we get

$$d^\alpha (\Omega^2)^\alpha = \left\{(dq)^\alpha - \frac{\partial^\alpha (H_1, H2)}{\partial^\alpha (p, r)}(dt)^\alpha\right\} \wedge \left\{(dp)^\alpha - \frac{\partial^\alpha (H_1, H2)}{\partial^\alpha (q, r)}(dt)^\alpha\right\} \wedge \left\{(dr)^\alpha - \frac{\partial^\alpha (H_1, H2)}{\partial^\alpha (q, p)}(dt)^\alpha\right\}. \tag{16}$$

Now we get the generalized Pfaffian equations

$$(dq)^\alpha - \frac{\partial^\alpha (H_1, H_2)}{\partial^\alpha (p, r)}(dt)^\alpha = 0 \tag{17}$$

$$(dp)^\alpha - \frac{\partial^\alpha (H_1, H_2)}{\partial^\alpha (q, r)}(dt)^\alpha = 0 \tag{18}$$

$$(dr)^\alpha - \frac{\partial^\alpha (H_1, H_2)}{\partial^\alpha (q, p)}(dt)^\alpha = 0. \tag{19}$$

By analogy of integer order these equations are called Fractional Nambu equations.

6 Example

As an example we consider a coupling two symmetric tops. The coupling introduced is proportional to the r component. In the following equations are the generalized Nambu functions corresponding to our example, namely

$$H_1 = \frac{1}{2}(q_1^2 + p_1^2 + r_1^2) + \frac{1}{2}(q_2^2 + p_2^2 + r_2^2), \tag{20}$$

$$H_2 = \frac{1}{2}\left(\frac{q_1^2}{I_{q_1}} + \frac{p_1^2}{I_{p_1}} + \frac{r_1^2}{I_{r_1}}\right) + \frac{1}{2}\left(\frac{q_2^2}{I_{q_2}} + \frac{p_2^2}{I_{p_2}} + \frac{r_2^2}{I_{r_2}}\right) + cr_1 r_2. \tag{21}$$

Where q_i, p_i, r_i $(i = 1, 2)$ angular momenta of the tops in their respective body coordinates and $I_{q_i}, I_{p_i}, I_{r_i}$ are the moments of inertia about the q_i, p_i, r_i axes respectively. The constant

c depends on the initial orientation on the tops in the laboratory frame. Using (17), (18) and (19) we obtain

$$(dq_i)^\alpha - \frac{\partial^\alpha (H_1, H_2)}{\partial^\alpha (p_i, r_i)}(dt)^\alpha = 0, \quad i = 1, 2 \tag{22}$$

$$(dp_i)^\alpha - \frac{\partial^\alpha (H_1, H_2)}{\partial^\alpha (q_i, r_i)}(dt)^\alpha = 0, \quad i = 1, 2 \tag{23}$$

$$(dr_i)^\alpha - \frac{\partial^\alpha (H_1, H_2)}{\partial^\alpha (q_i, p_i)}(dt)^\alpha = 0, \quad i = 1, 2. \tag{24}$$

It follows easily generalized coupling two symmetric tops equations

$$\begin{aligned}(dq_1)^\alpha - \Bigg\{ &\frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{p_1^{2-\alpha} r_1^{2-\alpha}}{I_{r_1}} + \frac{c}{2}\frac{\Gamma(3)}{\Gamma(3-\alpha)}\frac{\Gamma(2)}{\Gamma(2-\alpha)} r_2 r_1^{1-\alpha} p_1^{2-\alpha} \\ &- \frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{p_1^{2-\alpha} r_1^{2-\alpha}}{I_{p_1}} \Bigg\}(dt)^\alpha = 0,\end{aligned} \tag{25}$$

$$\begin{aligned}(dp_1)^\alpha - \Bigg\{ &\frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{q_1^{2-\alpha} r_1^{2-\alpha}}{I_{r_1}} + \frac{c}{2}\frac{\Gamma(3)}{\Gamma(3-\alpha)}\frac{\Gamma(2)}{\Gamma(2-\alpha)} r_2 r_1^{1-\alpha} q_1^{2-\alpha} \\ &- \frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{q_1^{2-\alpha} r_1^{2-\alpha}}{I_{q_1}} \Bigg\}(dt)^\alpha = 0,\end{aligned} \tag{26}$$

$$(dr_1)^\alpha - \left\{ \frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{q_1^{2-\alpha} p_1^{2-\alpha}}{I_{p_1}} - \frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{p_1^{2-\alpha} q_1^{2-\alpha}}{I_{q_1}} \right\}(dt)^\alpha = 0. \tag{27}$$

Also we can obtain for q_2, p_2 and r_2 the same equations as follows:

$$\begin{aligned}(dq_2)^\alpha - \Bigg\{ &\frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{p_2^{2-\alpha} r_2^{2-\alpha}}{I_{r_2}} + \frac{c}{2}\frac{\Gamma(3)}{\Gamma(3-\alpha)}\frac{\Gamma(2)}{\Gamma(2-\alpha)} r_1 r_2^{1-\alpha} p_2^{2-\alpha} \\ &- \frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{p_2^{2-\alpha} r_2^{2-\alpha}}{I_{p_2}} \Bigg\}(dt)^\alpha = 0,\end{aligned} \tag{28}$$

$$\begin{aligned}(dp_2)^\alpha - \Bigg\{ &\frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{q_2^{2-\alpha} r_2^{2-\alpha}}{I_{r_2}} + \frac{c}{2}\frac{\Gamma(3)}{\Gamma(3-\alpha)}\frac{\Gamma(2)}{\Gamma(2-\alpha)} r_1 r_2^{1-\alpha} q_2^{2-\alpha} \\ &- \frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{q_2^{2-\alpha} r_2^{2-\alpha}}{I_{q_2}} \Bigg\}(dt)^\alpha = 0,\end{aligned} \tag{29}$$

$$(dr_2)^\alpha - \left\{ \frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{q_2^{2-\alpha} p_2^{2-\alpha}}{I_{p_2}} - \frac{1}{4}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]^2 \frac{p_2^{2-\alpha} q_2^{2-\alpha}}{I_{q_2}} \right\}(dt)^\alpha = 0. \tag{30}$$

Remark The standard case corresponds to $\alpha = 1$.

7 Conclusion

The fractional dynamics started to play an important role in various fields of science and engineering. The fractional mechanics was found suitable for describing the phenomena involving friction. The approaches to generalize the classical mechanics are different, one is based on fractional forms and another one is based on the fractional generalization of the classical mechanics, namely by using the fractional variational principles and the fractional Legendre transformation. In this manuscript we used the method based of fractional differential forms and the fractional exterior derivative of the fractional 2-form. The generalized Pfaffian equations are obtained. The example represents a coupling of two symmetric tops. The corresponding Pfaffian equations are obtained and in the limit $\alpha = 1$ the classical case is obtained.

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